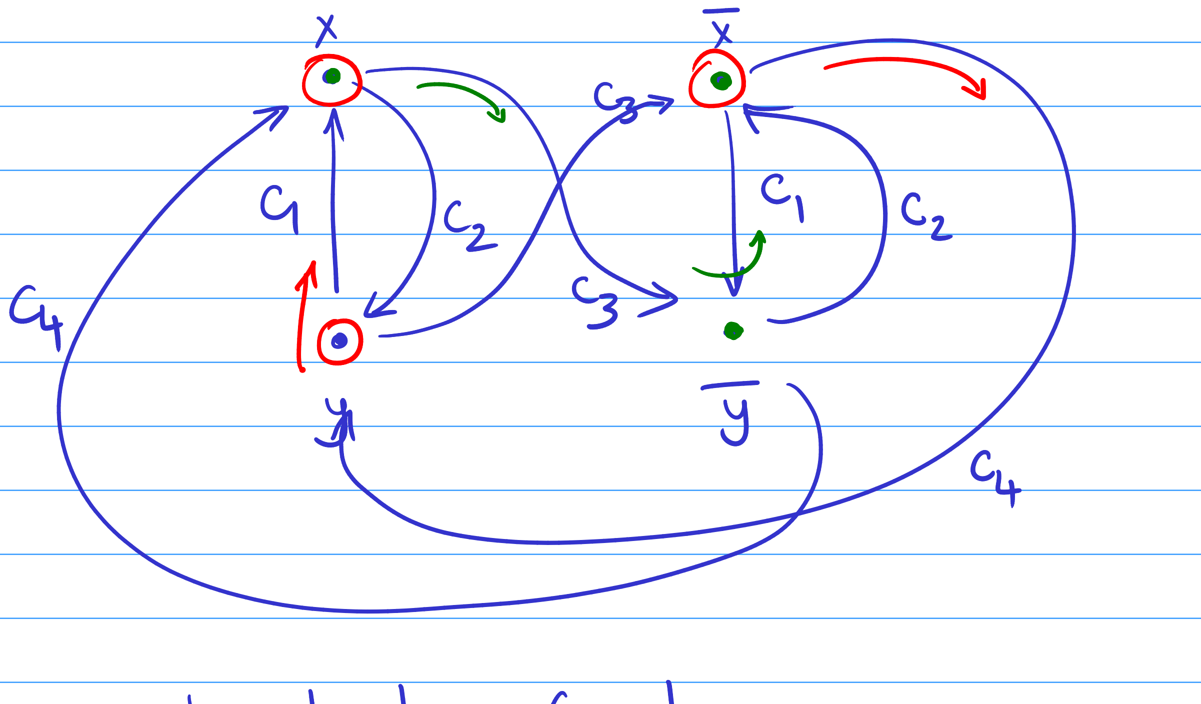
Lecture 6

Checking satisfiability of 2-CNF formulas

Ex: $\underbrace{(x \vee \bar{y})}_{C_1}$ and $\underbrace{(\bar{x} \vee y)}_{C_2}$ and $\underbrace{(\bar{x} \vee \bar{y})}_{C_3}$ and $\underbrace{(x \vee y)}_{C_4}$

$\bar{x} \Rightarrow \bar{y}$ $\quad x \Rightarrow y$ $\quad x \Rightarrow \bar{y}$ $\quad \bar{x} \Rightarrow y$

$y \Rightarrow x$ $\quad \bar{y} \Rightarrow \bar{x}$ $\quad y \Rightarrow \bar{x}$ $\quad \bar{y} \Rightarrow x$



Implication Graph.

- <u>Claim</u>: From the implication graph, G_φ, we can reconstruct the original formula φ.

$$x \longrightarrow y$$

$x \Rightarrow y \quad \bar{x} \vee y$

- <u>Therefore</u>, satisfiability reduces to <u>some</u> graph theoretic property of G_φ

- <u>Claim</u>: φ is unsatisfiable iff for some variable x,

x & $\bar{x}$ occur in the same SCC in G_φ.

$$\begin{tikzcd} x \arrow[r, squiggly, bend left] & \bar{x} \arrow[l, squiggly, bend left] \end{tikzcd}$$

Proof ($\Leftarrow$) Assume not.
Let a be a satisfying assignment of φ

Case 1: $a_x = \text{true}$.

We know that there is a path from

$$x \rightsquigarrow \bar{x}$$

$$x \rightsquigarrow \overset{\ell}{\bullet} \longrightarrow \bar{x}$$

There is some clause $(\ell \Rightarrow \bar{x})$ in φ.

$\underbrace{\text{By induction, literal } \ell \text{ is true} \qquad \text{But } \bar{x} \text{ is false}}$

So this clause is false under assignment a.

This violates the assumption that a was a satisfying assignment of the formula.

Case 2: Say $a_x = \text{false}$

$$\bar{x} \rightsquigarrow x$$

$$\bar{x} \rightsquigarrow \ell \longrightarrow x$$

$\ell \Rightarrow x$

Proof: ($\Rightarrow$) If φ is unsat, then for some variable x, x & $\bar{x}$ occur in the same scc.

We need to show.

Proof by contrapositive.

We want to show "If a then b"

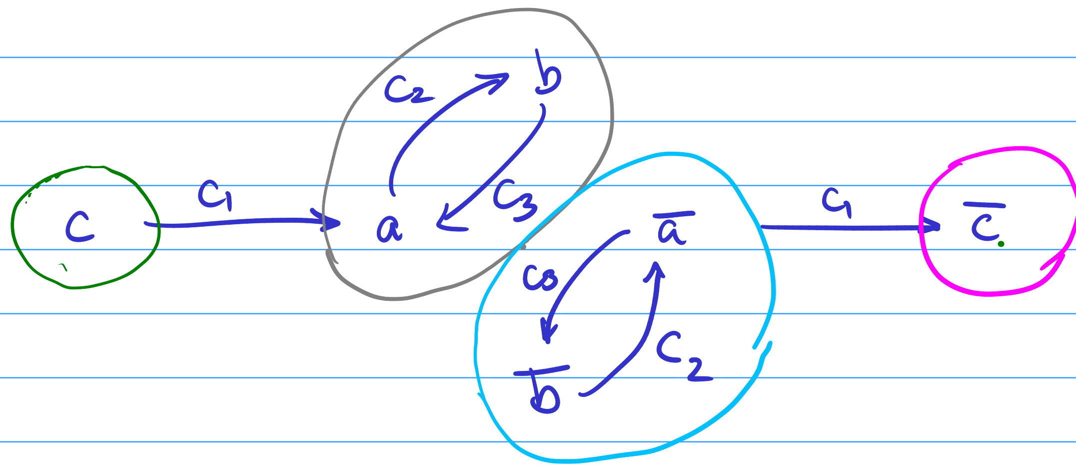
We instead show "If not b then not a".

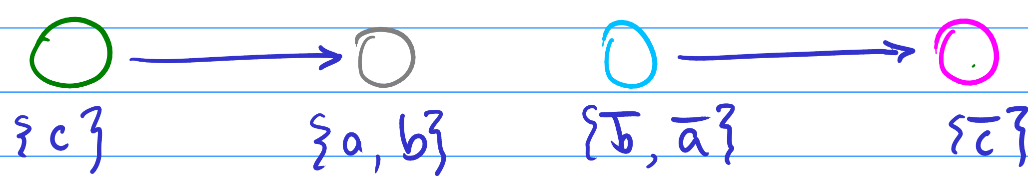
We will show:

If $\forall$ variables x, x & $\bar{x}$ don't occur in the same scc

then φ is satisfiable.

Ex: $\overbrace{(c \Rightarrow a)}^{c_1}$ and $\overbrace{(a \Rightarrow b)}^{c_2}$ and $\overbrace{(b \Rightarrow a)}^{c_3}$





Condensation graph:

1. One node for each scc
2. Edge scc $C_1 \longrightarrow$ scc C_2

iff $C_1 \neq C_2$ & $\exists\ c_1 \in C_1$ $\quad c_2 \in C_2$

s.t $\quad c_1 \longrightarrow c_2$ edge exists in the original graph.

Claim: The condensation graph is acyclic.
(directed)

Idea: You can do a topological sort of a DAG!

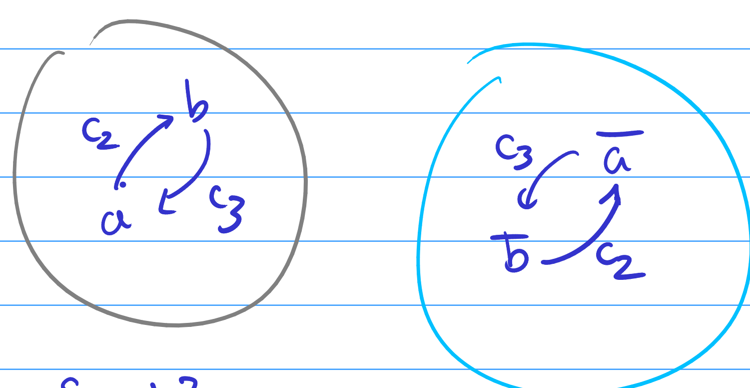
① $\{c\} \longrightarrow \{a, b\}$ $\quad \{\bar{b}, \bar{a}\} \longrightarrow \{\bar{c}\}$

Then, all literals here become false - - - - - - - - - - - - - set all literals to true.

$\bar{c}$ = true , c = false

And this SCC

Throw away this SCC

② 

$\bar{a}$ = true , a = false ..

$\bar{b}$ = true , b = false ...

$\{a, b\}$ $\quad \{\bar{b}, \bar{a}\}$

~~Then,~~
Throw away both SCC's.

Same as setting all literals here to false - - - set all literals here to true

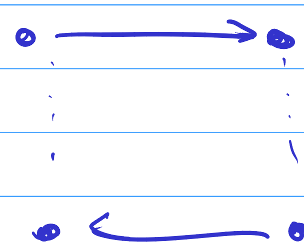
③ ~~Partial~~ assignment so far

$$\boxed{\begin{array}{l} a = \text{false} \\ b = \text{false} \\ c = \text{false} \end{array}}$$

It is in fact a complete assignment.

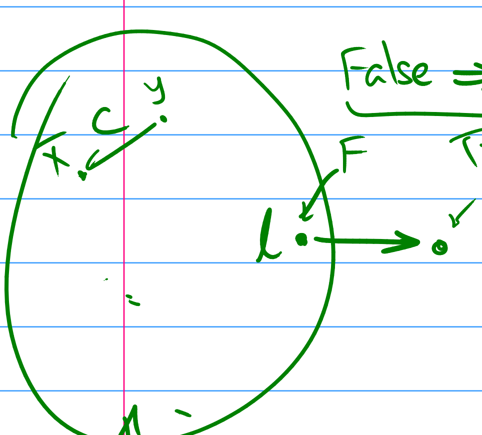
<u>Claim</u>: This assignment satisfies the formula.

<u>Yifei's observation</u>: The graph G_φ is skew symmetric.



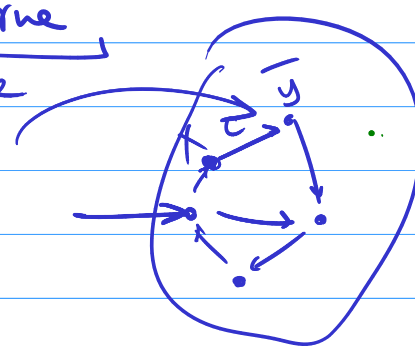
$\bar{l'} \Rightarrow \bar{l}$ | $\overset{T}{\overline{F \Rightarrow F}}$

$l \Rightarrow l'$ | $\overset{T}{\overline{T \Rightarrow T}}$



$\underbrace{\text{False} \Rightarrow \underline{\quad\quad}}_{\text{True}}$

$\underbrace{l \Rightarrow \text{true}}_{\text{True}}$



Then I set everything here to false

If I say everything is true here

a	b	$a \Rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

CNF : Conjunctive Normal Form

Clause

[(___ or ___ or ___) and
(___ or ___) and
(___ or ___ or ___ or ___) and
⋮

Literal

x $\bar{x}$

- Conflict Driven Clause Learning (CDCL)

Heuristic optimization on top of $DPLL_{up}$.

$(x_1 \vee x_4)$ and

$(x_1 \vee \overline{x_3} \vee \overline{x_8})$ and

$(x_1 \vee x_8 \vee x_{12})$ and

$(x_2 \vee x_{11})$ and

$(\overline{x_7} \vee \overline{x_3} \vee x_9)$ and

$(\overline{x_7} \vee x_8 \vee \overline{x_9})$ and

$(x_7 \vee x_8 \vee \overline{x_{10}})$ and

$(x_7 \vee x_{10} \vee \overline{x_{12}})$

$(x_1 \vee x_4)$ and

$(x_1 \vee \bar{x}_3 \vee \bar{x}_8)$ and

$(x_1 \vee x_8 \vee x_{12})$ and

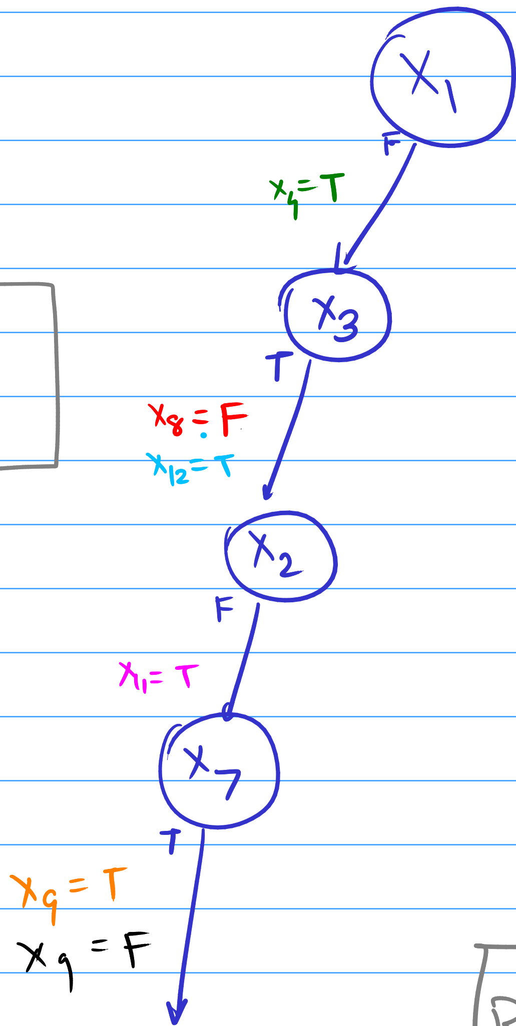
$(x_2 \vee x_{11})$ and

$(\bar{x}_7 \vee \bar{x}_3 \vee x_9)$ and

$(\bar{x}_7 \vee x_8 \vee \bar{x}_9)$ and

$(x_7 \vee x_8 \vee \bar{x}_{10})$ and

$(x_7 \vee x_{10} \vee \bar{x}_{12})$



Resolution

If] $(\bar{x}_7 \vee \bar{x}_3 \vee x_9)$ and $(\bar{x}_7 \vee x_8 \vee \bar{x}_9)$

(If x_9 is false) $\bar{x}_7 \vee \bar{x}_3$

(If x_9 is true) $\bar{x}_7 \vee x_8$

Then] $(\bar{x}_7 \vee \bar{x}_3 \vee \bar{x}_7 \vee x_8)$

$(x_1 \vee x_4)$ and

$(x_1 \vee \overline{x_3} \vee \overline{x_8})$ and

$(x_1 \vee x_8 \vee x_{12})$ and

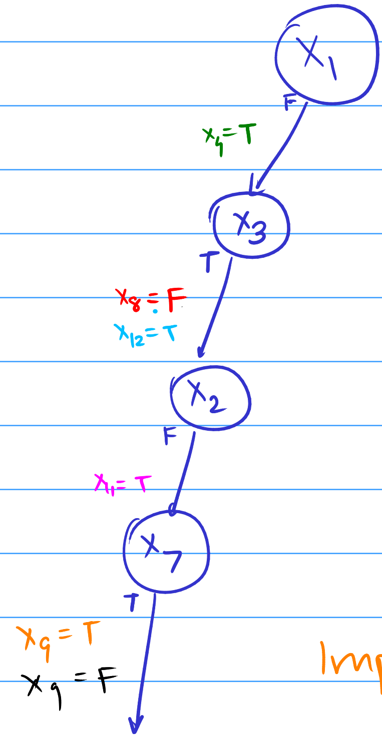
$(x_2 \vee x_{11})$ and

$(\overline{x_7} \vee \overline{x_3} \vee x_9)$ and

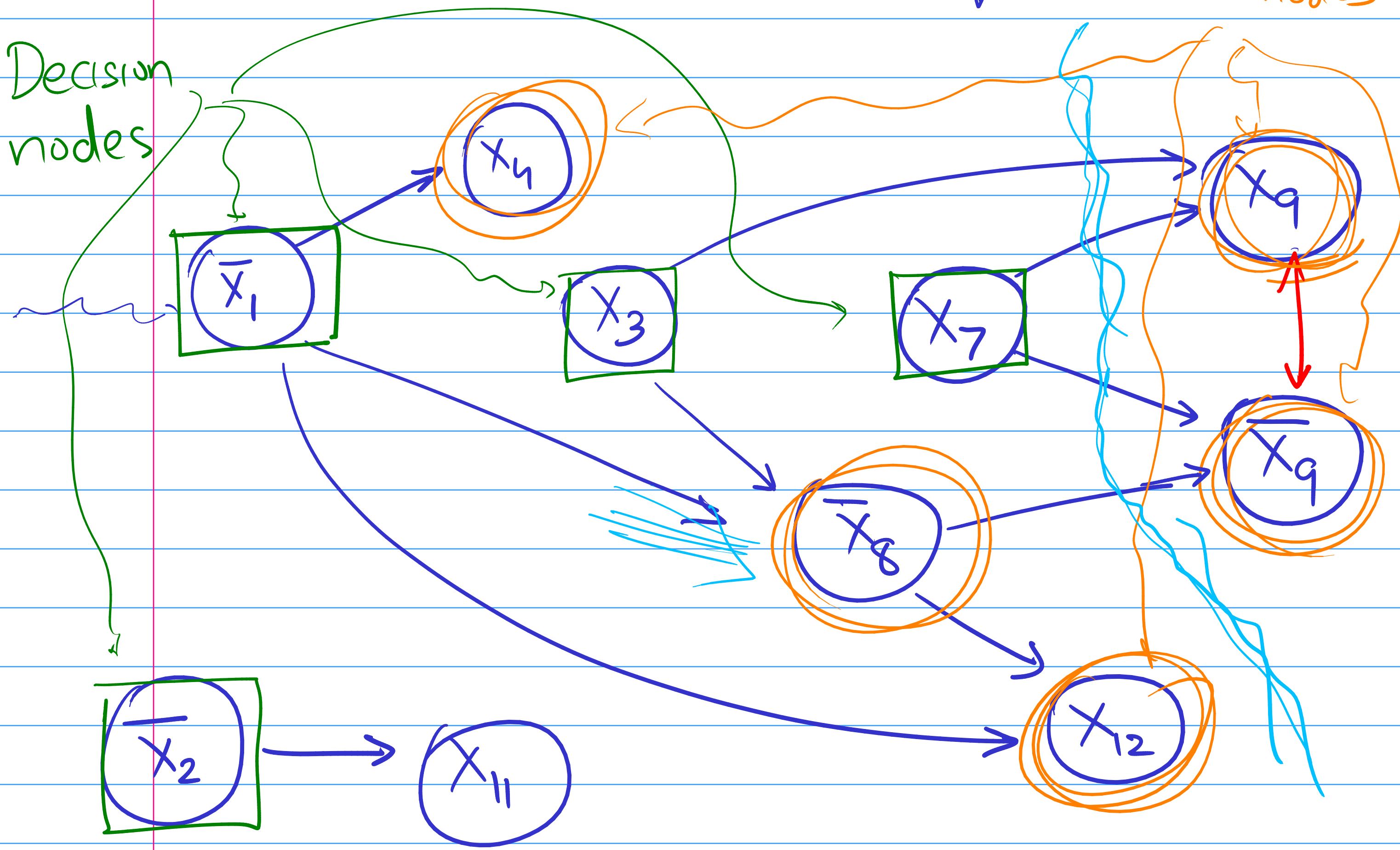
$(\overline{x_7} \vee x_8 \vee \overline{x_9})$ and

$(x_7 \vee x_8 \vee \overline{x_{10}})$ and

$(x_7 \vee x_{10} \vee \overline{x_{12}})$



Implication nodes



Implication Graph

Immediate conclusion

Possibility 1 : Never set x_3 & x_7 to be true together.

$\overline{x_3} \vee \overline{x_7}$ ← I suspect that this is not sound.

Possibility 2 : Don't repeat these voluntary choices.

$x_1 \vee x_2 \vee \overline{x_3} \vee \overline{x_7}$

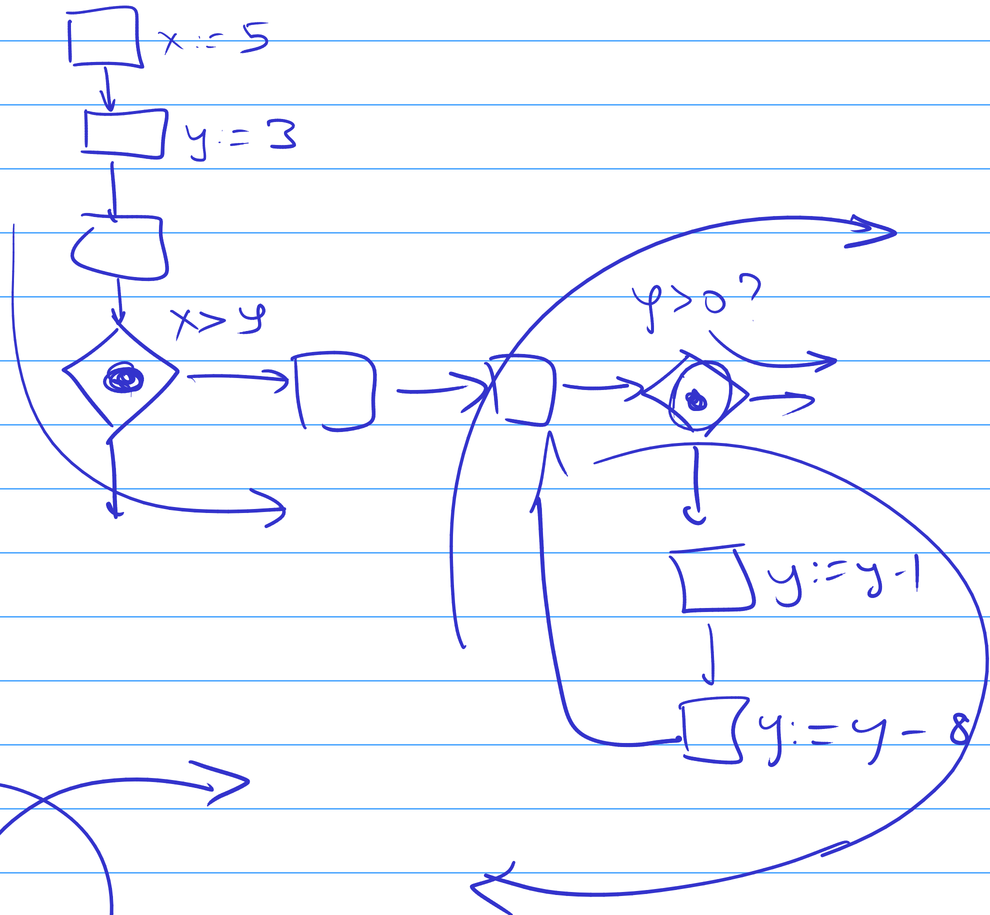
Possibility 3 : Never set x_3 & x_7 & $\overline{x_1}$ to be true together.

$x_1 \vee \overline{x_3} \vee \overline{x_7}$

Possibility 4 Don't repeat x_3 & x_7 & $\overline{x_8}$

$\overline{x_3} \vee \overline{x_7} \vee x_8$

If x_3 & x_7 & $\overline{x_8}$ then conflict guaranteed.


x := 5
y := 3
x > y
y > 0 ?
y := y - 1
y := y - 8