Lecture 8 : Introduction to SMT

- DPLL (T)
- Specific Ts : Difference logic

  LRA

  E / EUF
- Theory combination : The Nelson Oppen procedure

Solvers for Linear Real Arithmetic

Question . Do there exist real numbers x, y such that

Multiplication with constants is okay.

$x + 2y \leq 5$	$x + 2y \leq 5$
and $(x - 8y > 3)$	and $x - 8y > 3$
and not $(x + 4y < 8)$	and $x + 4y \geq 8$

①

$x + 2y \le 5$

and $x - 8y > 3$

and $x + 4y \ge 8$

②

$y \le -\frac{x}{2} + \frac{5}{2}$

$y < \frac{x}{8} - \frac{3}{8}$

$-\frac{x}{4} + 2 \le y$

$x - 8y > 3$

$x - 3 > 8y \qquad y < \frac{1}{8}x - \frac{3}{8}$

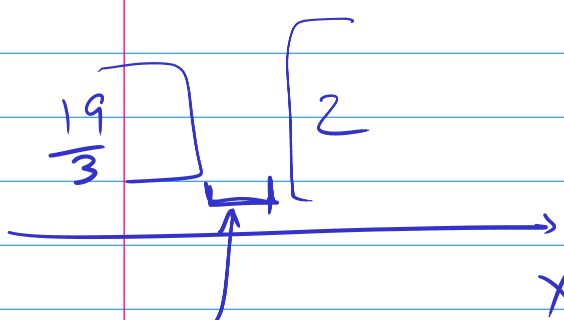
$y \ge -\frac{x}{4} + 2$

$\frac{x}{4} \le \frac{1}{2} \qquad x \le 2$

$2 + \frac{3}{8} < \frac{3x}{8}$

$19 < 3x$

$\frac{19}{3} < x$



$x < 4$

$3 < x$

③ Does $\exists x$ such that

$-\frac{x}{4} + 2 \le -\frac{x}{2} + \frac{5}{2}$

$-\frac{x}{4} + 2 < \frac{x}{8} - \frac{3}{8}$

3, in other words

Does $\exists x$ such that

$x \le 2$

$\frac{19}{3} < x$

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④ Is it the case that

$\frac{19}{3} < 2$

linear real constraints

Variable elimination : $\exists x \underline{\exists y} . \varphi_2$?

$\underline{\exists x}$ φ_1 ?

also: linear real

φ_0 ?

linear real

Strict upper bounds

$x < \_\_\_\_\_\_$

$x < \_\_\_\_\_\_$

$\vdots$

Non-strict upper bounds

$x \leq \_\_\_\_\_\_$

$x \leq \_\_\_\_\_\_$

$\vdots$

Non strict lower bounds

$\_\_\_\_\_\_ \leq x$

$\_\_\_\_\_\_ \leq x$

$\vdots$

Strict lower bounds

$\_\_\_\_\_\_ < x$

$\_\_\_\_\_\_ < x$

$$\left.\begin{array}{r} x < u_s \\ x \le u_{ns} \\ l_{ns} \le x \\ l_s < x \end{array}\right\} \Longrightarrow \begin{array}{l} l_{ns} < u_s \\ l_{ns} \le u_{ns} \\ l_s < u_s \\ l_s < u_{ns} \end{array}$$

<u>Question</u>: I called this an algorithm for LRA. Why is it not an algorithm for LIA?

For example:

- If $x \in \mathbb{R}$, $3 < x$ and $x < 4$ is sat

$$x = 3.5$$

- If $x \in \mathbb{Z}$, $3 < x$ and $x < 4$ is unsat

- For a simple decision procedure for LIA see chapter 4 of Sipser's textbook.

- ## Theory Solvers for Difference Logic

Every constraint is of the form $u - v \leq c$

or of the form $u - v < c$

Aside:

$u - v < c$ iff $u - v \leq c - 1$, if $u, v \in \mathbb{Z}$

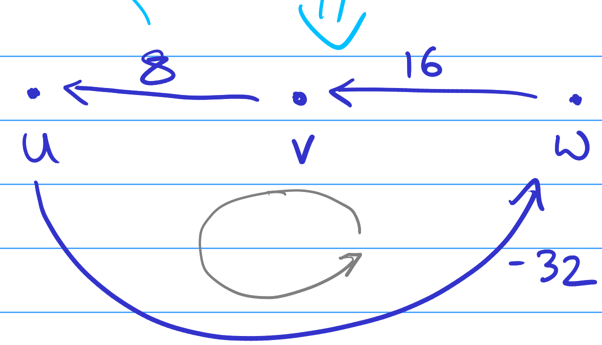
- **Example:**

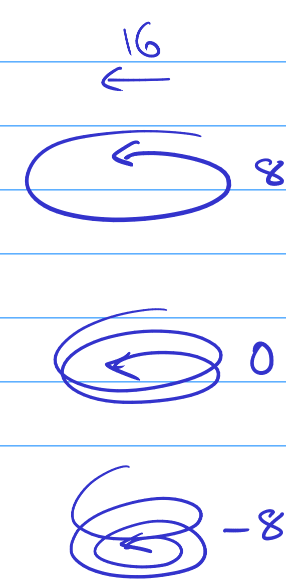
$u - v \leq 8$

$v - w \leq 16$

$u - w \geq 32$ $\quad$ $w - u \leq -32$

Each edge corresponds to some inequality





- From the difference graph, we can recover the original constraints
- So the constraints are satisfiable iff the graph has some property.
- What property could this be?

Proposal 1: Something about shortest paths

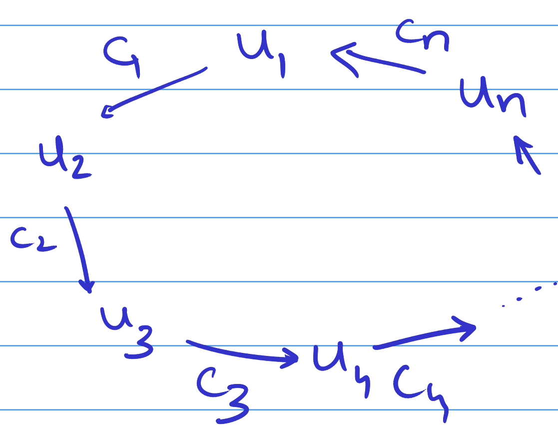
Observation 2: Negative weight cycle exists

Claim: The set of difference constraints φ is satisfiable $\boxed{\text{iff}}$ the corresponding difference graph G_φ does not contain any negative weight cycles

- If you accept the claim, simply implement any shortest path algorithm with support for negative weights. It will contain an escape hatch for negative weight cycles. Simply check if the escape hatch is triggered.

Proof ①: If negative weight cycle exists then formula is unsat.

Assume not. Look at the -ve weight cycle.



$$\left.\begin{aligned} \cancel{u_2} - u_1 &\le c_1 \\ u_3 - \cancel{u_2} &\le c_2 \end{aligned}\right] \quad u_3 - u_1 \le c_1 + c_2$$

$$u_4 - u_3 \le c_3 \quad\quad u_4 - u_1 \le c_1 + c_2 + c_3$$

$$\vdots$$

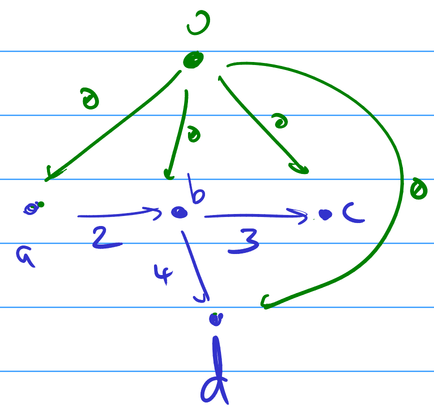
$$u_1 - u_n \le c_n$$

$$\cancel{u_1} - \cancel{u_1} \le \underbrace{\sum_i c_i}$$

Contradiction! So satisfying assignment cannot exist. $0 \le -ve$.

Proof, part 2: No negative weight cycle exists.

How to find satisfying assignment?



Set a = shortest path from o to a.

$b - a \leq 2$

$c - b \leq 3$

$d - b \leq 4$

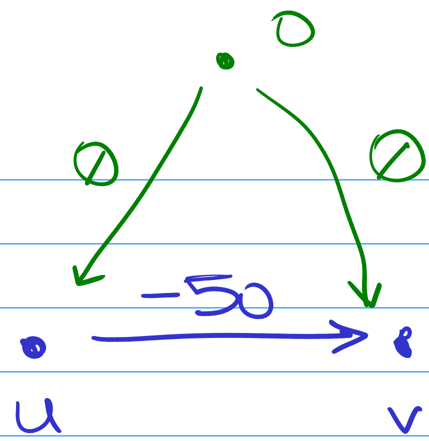
In this case, set all variables to 0.

- In other words, create a new variable o.
- For each old variable x, add the constraint $x - o \leq 0$ (i.e. $x \leq o$)
- Set x to shortest path from o to x.
- **Claim**: This is a satisfying assignment.

$v - u \le -50$

$u - 0 \le \emptyset$

$v - 0 \le \emptyset$



Claim: Adding the green constraints does not affect the satisfiability of the blue constraints

$u - 0 \le \emptyset \qquad \Big\| \qquad u \le 0$

$v - 0 \le \emptyset \qquad \Big\| \qquad v \le 0$

Proof: Pick any satisfying assignment of the original system.

Say $\boxed{u = 0, \quad v = -75}$ → This satisfies the blue constr.

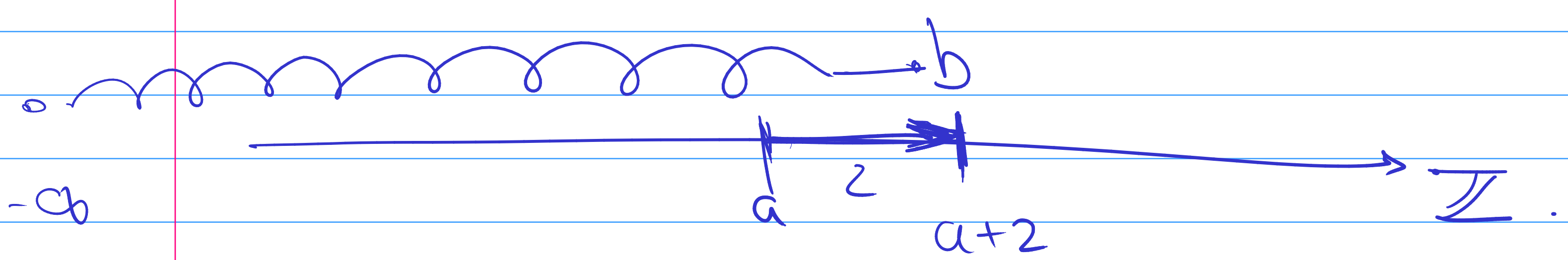
Pick $0 = 100$ (very. v. large num)

By construction, our choice of o should satisfy the green constraints.

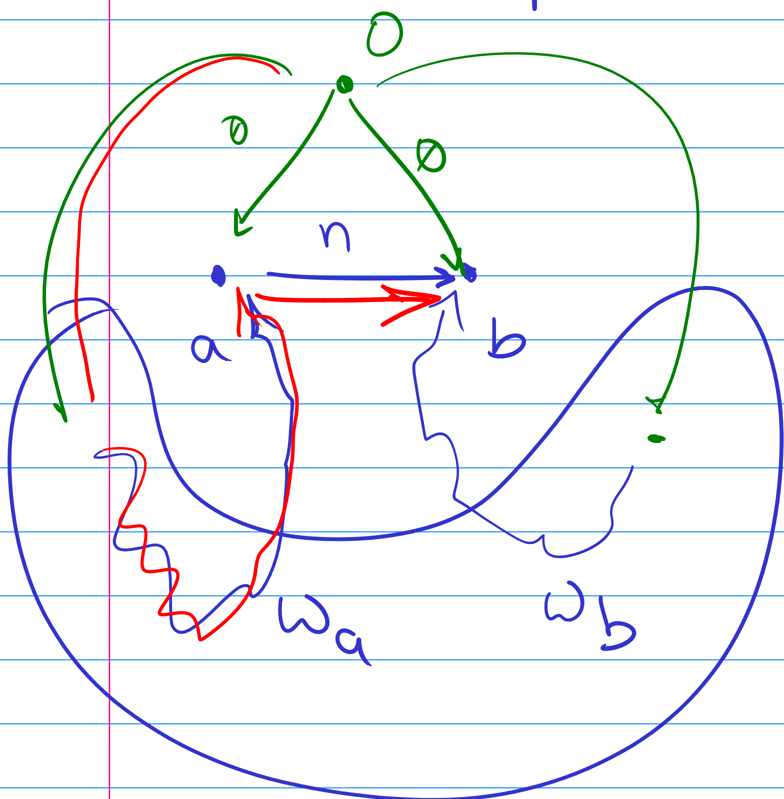
Consider some edge

$$a \xrightarrow{2} b$$

$$b - a \leq 2$$



- Assume that the shortest paths do not yield a satisfying assignment.



$$\underbrace{b - a \le n}$$

Assume that this constraint is violated.

So $w_b - w_a \not\le n$.

$$w_b \le w_a + n$$