Lecture 11

Nelson Oppen Procedure

Claim: If NOP claims "unsat", then φ is actually unsat.

(Soundness).

(Completeness)

Claim: If NOP says "sat", then φ is actually sat.

Look at the assignment $\{x \mapsto \_\_$
$y \mapsto \_\_$
$\dots$
$\}$

Observe that it satisfies φ.

Claim: If NOP says "sat", then φ is actually sat.

Claim': If φ is actually unsat then NOP says "unsat".

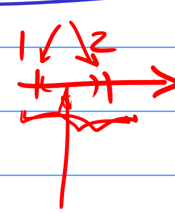
Claim: The NOP algorithm always terminates.

Proof: Only finitely many equality constraints can be enforced on n variables.

Ex: Consider LIA. ← LIA is not a convex theory.

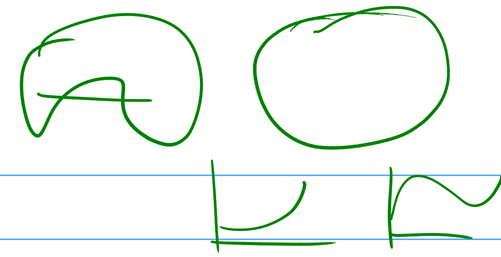
Thm: $\forall x.\ 1 \le x \ \wedge\ x \le 2 \Rightarrow x = 1 \text{ or } x = 2$

Does this mean that either



$\forall x\ (1 \le x \le 2 \Rightarrow x = 1)$: Prop 1

or $\forall x\ (1 \le x \le 2 \Rightarrow x = 2)$: Prop 2



A theory is convex if

conjunctive formula

whenever $\varphi \Rightarrow x_1=y_1 \text{ or } x_2=y_2 \text{ or } \cdots \text{ or } x_n=y_n$

then, at least one of the following is true:

$\boxed{\varphi \Rightarrow x_1=y_1}$

or $\varphi \Rightarrow x_2=y_2$

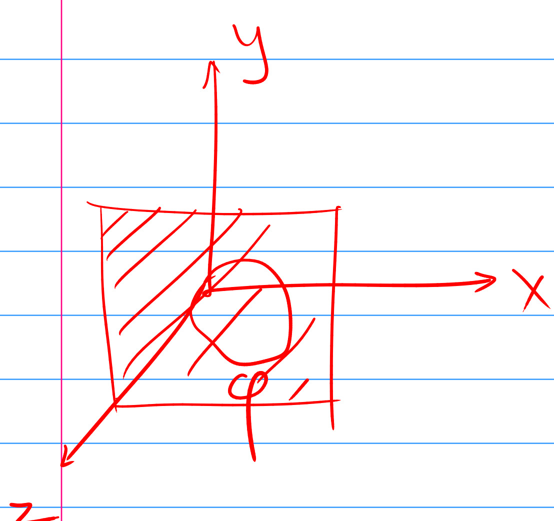
or ...

or $\varphi \Rightarrow x_n=y_n$

LRA is a convex theory.

$\underline{w=u \wedge x=y \wedge y=z} \Rightarrow x=z$ or $\underline{w=x}$

$w=u \wedge x=y \wedge y=z \Rightarrow x=z$



$(1 \leq x \leq 2 \Rightarrow x = 1)$ or $(1 \leq x \leq 2 \Rightarrow x = 2)$

$\not\Leftarrow$

Proposition 1: $\forall x \in \mathbb{Z}$ if $1 \leq x \leq 2$

then $x = 1$ or $x = 2$

Prop 2: $\forall x \in \mathbb{R}$ if $1 \leq x \leq 2$

then $x = 1$ or $x = 2$

$\exists x .\ 1 \leq x \leq 2$ but not $1 = x$

$\boxed{T_1}$ $\boxed{T_2}$

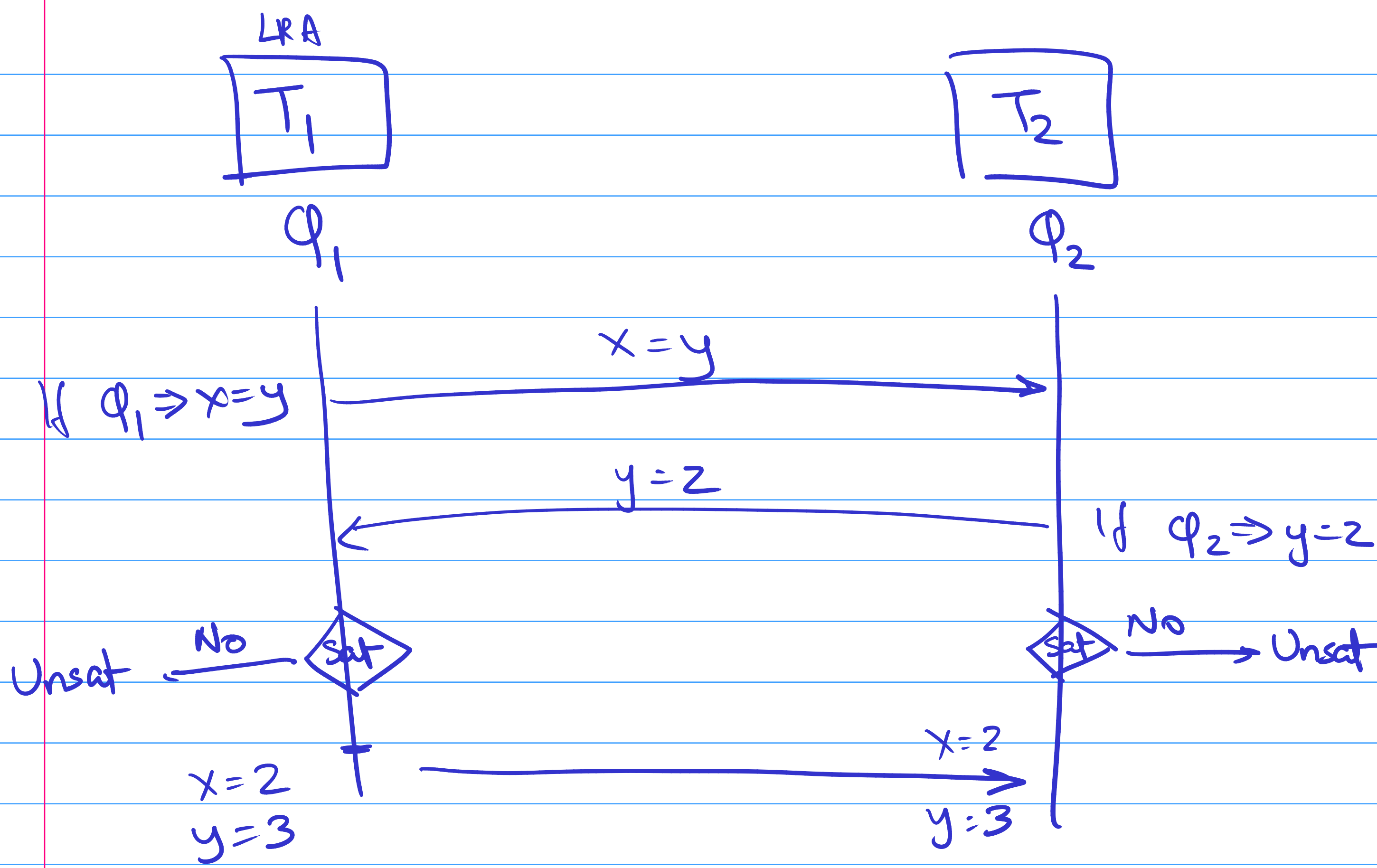
φ_1 φ_2

$\boxed{x = y \text{ or } y = 2}$

$x = y$ and $y = 2$

$\boxed{\varphi_1 \Rightarrow x = y}$

$\boxed{\varphi_1 \Rightarrow y = 2}$



LRA
T_1
T_2
φ_1
φ_2
$x=y$
If $\varphi_1 \Rightarrow x=y$
$y=2$
If $\varphi_2 \Rightarrow y=2$
Unsat
No
sat
sat
No
Unsat
$x=2$
$y=3$
$x=2$
$y=3$

Unit 2: Program Synthesis

Input Output Examples

If $x=1$ then $f(x)=8$

If $x=2$ then $f(x)=12$

If $x=1$ then
$f(x)=$ true

If $x=2$ then
$f(x)=$ false

Functional constraints

If $y=0$ then $f(x\ y)=x$

If $x=0$ then $f(x\ y)=y$

$$\forall x\ y.\ f(x\ y) \geq x$$
$$\text{and } f(x\ y) \geq y$$
$$\text{and } \big(f(x\ y)=x \text{ or } f(x\ y)=y\big)$$

$\forall \vec{x} \quad \varphi(\vec{x})$

$\exists \vec{x}. \quad \varphi(\vec{x})$

Synthesis Problem VI

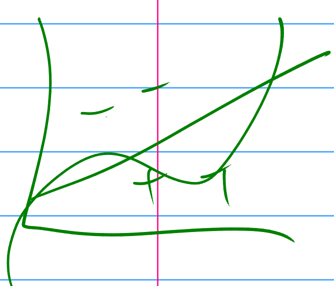
$$\boxed{\exists f.\ \forall \vec{x}.\ \varphi(f\ \vec{x})\ ?}$$

- $\exists f$? $\forall x \quad x=1 \Rightarrow f(x)=2$

  and $\quad x=2 \Rightarrow f(x)=8$

$$f(x) = 6x - 4$$

"Optimum synthesis"



Can we determine the smallest function that satisfies the constraints?

$\exists f \cdot \forall \vec{x} \cdot \; \varphi(f \; \vec{x})$?

- $\exists f \cdot \forall x \; (x = 1 \Rightarrow f(x) = 2)$ ← synth form

- $\underbrace{\forall x \cdot \exists y}$ $(x = 1 \Rightarrow y = 2)$ ← F.O. form

- $\exists f \cdot \forall x \cdot \; f(x) \leq f(x+1)$ ← synth form

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PSPACE-complete.

TQBF = (True) Quantified Boolean Formulas

Polynomial hierarchy

⋮

Σ^2 $\quad \exists x_2 \cdot \forall x_1 \cdot \exists x_0 \; \varphi \qquad \forall x_2 \cdot \exists x_1 \cdot \forall x_0 \cdot \varphi \; \Pi^2$

Σ^1 $\quad \exists x_1 \; \forall x_0 \; \varphi \qquad \forall x_1 \cdot \exists x_0 \cdot \; \varphi \quad \Pi^1$

Program synthesis

sat NPC $\quad \exists x_0 \cdot \varphi \qquad \forall x_0 \cdot \varphi$ $\quad$ valid coNPc

Syntax Guided Synthesis

Synthesis Problem v2

Grammar G (context free?) ← Syntax

$\exists f \sim G \,.\, \forall \vec{x} \quad \varphi(f \ \vec{x})$

$$\begin{aligned}\text{Start} ::= \ & 0 \mid 1 \mid x \mid y \\ & \mid \text{Start}_1 + \text{Start}_2 \mid \text{Start}_1 - \text{Start}_2 \\ & \mid \text{if}(\text{Start}_1 \leq \text{Start}_2) \text{ then } \\ & \qquad \text{Start}_1 \text{ else } \text{Start}_2\end{aligned}$$

(define-fun f ((a Int) (b Int)) Int (ite (>= (+ a (* (- 1) b)) 1) b a))

-b

if (a - b ≥ 1) then b else a

if a ≥ b then b else a.