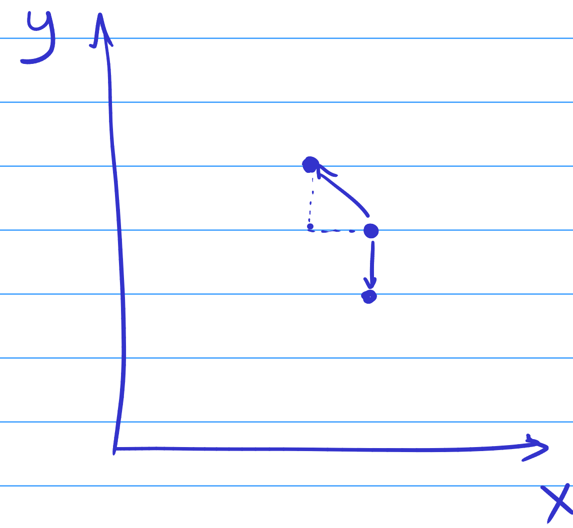
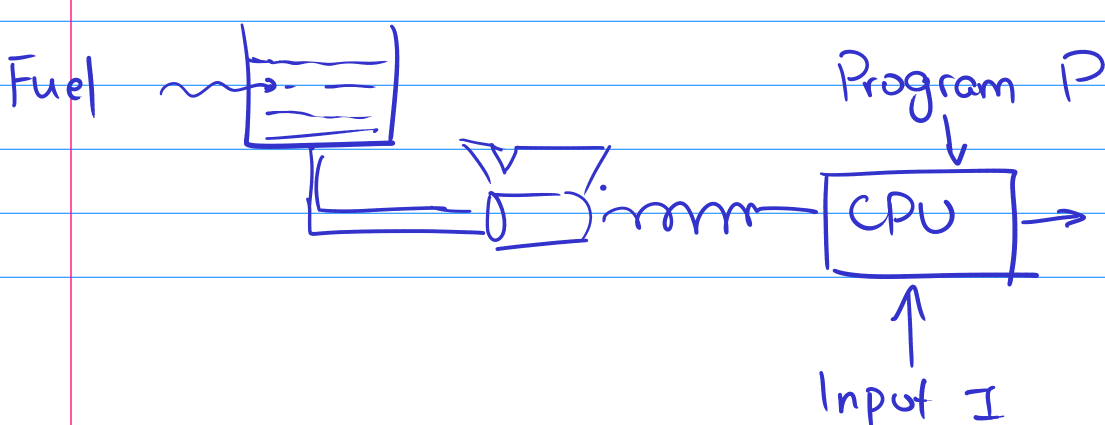
Lecture 21 : Proving Program Termination

Ex

```
x := input()
y := input()
while x>0 and y>0
    if input() = 1 then
        x := x-1
        y := y+1
    else: y := y-1
```



Analogy of a fuel tank



Requirements for f:

① $f(x\ y) \gneq f(x-1\ \ y+1)$

② $f(x\ y) \gneq f(x\ \ y-1)$

③ $x > 0$ and $y > 0 \Rightarrow f(x\ y) \gneq 0$

Guess: Amount of fuel needed is some linear function of x & y.

$$f(x\ y) = ax + by, \text{ for some } a, b.$$

① $ax + by \gneq a(x-1) + b(y+1)$

② $ax + by \gneq ax + b(y-1)$

③ $\underbrace{x > 0 \ \& \ y > 0 \Rightarrow ax + by > 0}$

$a \geq 0$ and $b \geq 0$

$x = 1$ and $y > 0 \Rightarrow ax + by > 0 \Leftrightarrow a + by > 0$

① $\cancel{ax} + \cancel{by} \not\geq a(x-1) + b(y+1)$

$$\not\geq \cancel{ax} - a + \cancel{by} + b$$

$a \not\geq b$

② $\cancel{ax} + \cancel{by} \not\geq \cancel{ax} + b(\cancel{y}-1)$

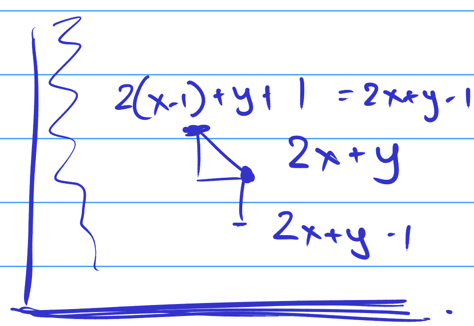
$0 \not\geq -b$

$b \not\geq 0$.

<u>Guess</u>: $b=1$ $a=2$

$$f(x\ y) = 2x+y$$

<u>Claim</u>: At all points, the computation will run for at most $2x+y$ steps.



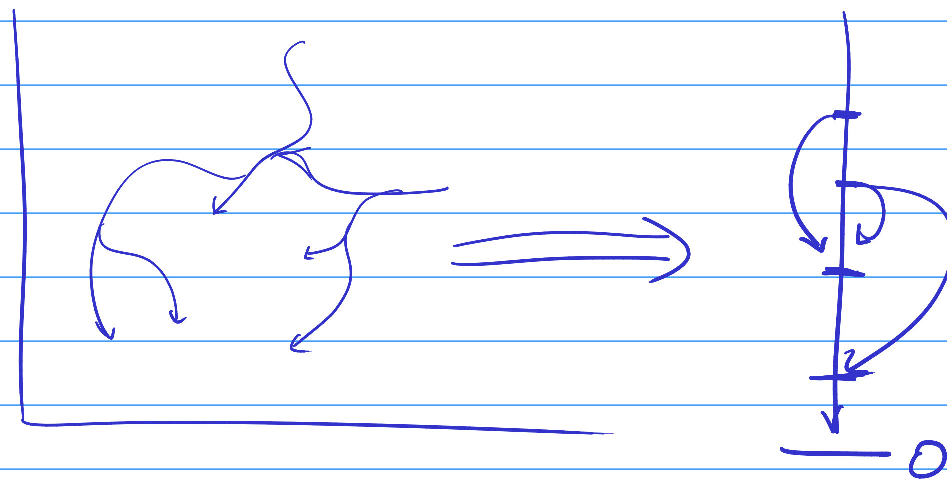
Reflection: We came up with a (linear) ranking function $f: Q \to \mathbb{N}$

We assume linearity only to make our guessing games easier.

such that:

①, ② whenever $q \to q'$ $\quad f(q) \geq f(q')$

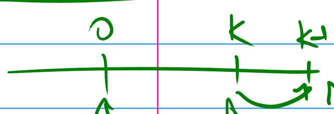
③ whenever $q \to q'$ $\quad f(q) \geq 0$



Reflection: $f: Q \to (\mathbb{N}, \leq)$ works as a well-order

$f: Q \to (\mathbb{R}, \leq)$ does not work as a well-order.

Induction



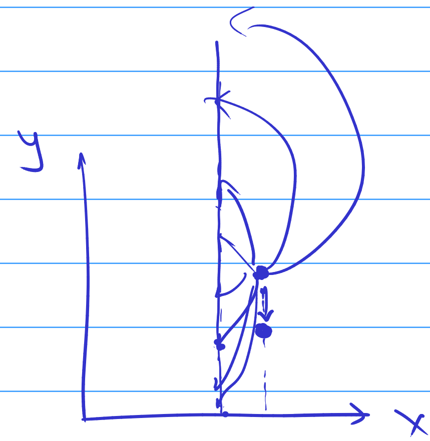
then φ also holds here.

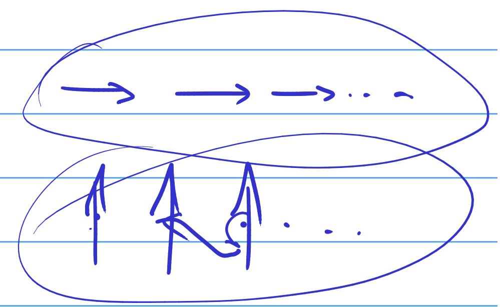
if φ holds here

show that φ holds here

<u>Ex</u>:

```
x := input ()
y := input ()
while x>0 and y>0
    if input () = 1 then
        x := x-1
        y := input ()
    else: y := y-1
```





<u>Ex</u>:

```
x := input ()
while x>0
    x := x-1
```

Arbitrarily long computations vs.

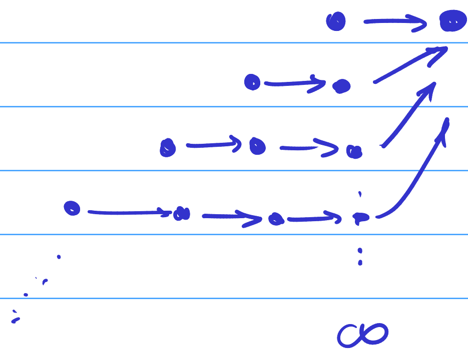
$\forall n, \exists$ input

$q_0 \to q_1 \to q_2 \to \cdots \to q_n$

Fin!

Infinite computations

$\exists$ input . $q_0 \to q_1 \to q_2 \to q_3 \to \cdots\cdots \infty$



Requirements for a linear ranking function

① $\forall x\ y,\ y'$ $\quad f(x\ y) \gneq f(x-1\ \ y')$

② $\forall x\ y$ $\quad f(x\ y) \gneq f(x\ \ y-1)$

③ $\forall x\ y$ $\quad x>0$ and $y>0 \Rightarrow f(x\ y) \gneq 0$

<u>Assume</u> $f(x\ y) = ax+by$

① $ax+by \gneq a(x-1)+by'$

$by \gneq -a+by'$

$a \gneq b(y'-y)$

There is no such a.

Conclusion: Sometimes, assuming that the ranking function is of the form

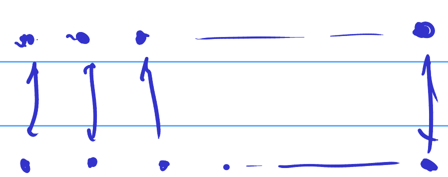
$$f : \mathbb{Q} \to \mathbb{N}$$

is insufficient

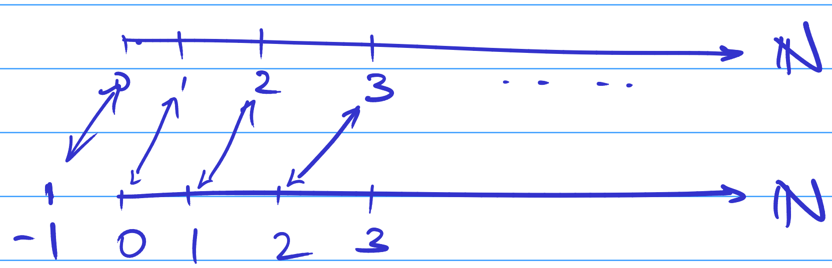
Aside: Cardinality of sets.

Countably infinite vs uncountably infinite sets

$\mathbb{N}$, $\mathbb{Q}$ $\quad \aleph_0 \quad\quad\quad\quad \mathbb{R}$



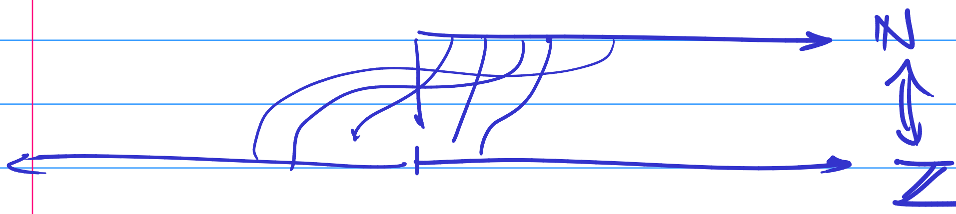
$$\aleph_0 = 1 + \aleph_0 .$$
$$= \aleph_0 + 1$$
$$= \aleph_0 + \aleph_0$$



$\mathbb{N}$

bijection exists

$\mathbb{N} \cup \{-1\}$



From counting (cardinal #s) to ordering (ordinal #s)

Claim 1: $\forall n \in \mathbb{N}, \ \varphi(n)$.

Proof by induction: Show $\varphi(0)$

Show, $\forall n, \ \varphi(n) \Rightarrow \varphi(n+1)$.

Claim 2: $\forall n \in \mathbb{N} \cup \{-1\} \quad \varphi(n)$.

Proof by induction: Show $\varphi(-1)$

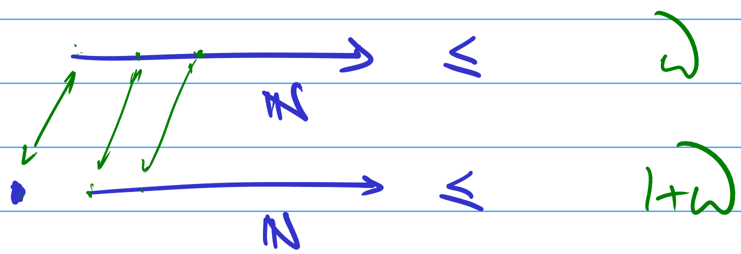
Show, $\forall n, \ \varphi(n) \Rightarrow \varphi(n+1)$

Claim 3: $\forall n \in \mathbb{N} \cup \{\infty\}$, $\varphi(n)$.

Proof attempt: Show $\varphi(0)$

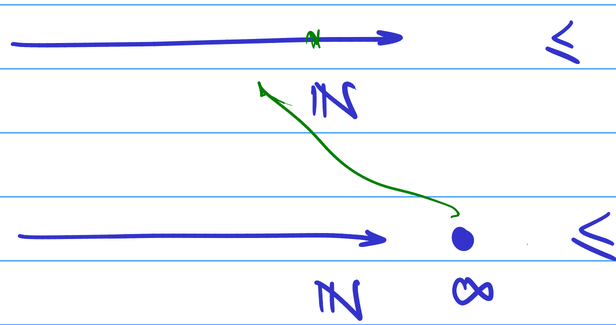
Show, $\forall n$, $\varphi(n) \Rightarrow \varphi(n+1)$.

Fail! We have not established $\varphi(\infty)$.



$\omega = 1+\omega$

Observation: There is an order preserving bijection between $(\mathbb{N}, \leq)$ & $(\mathbb{N} \cup \{-1\}, \leq)$



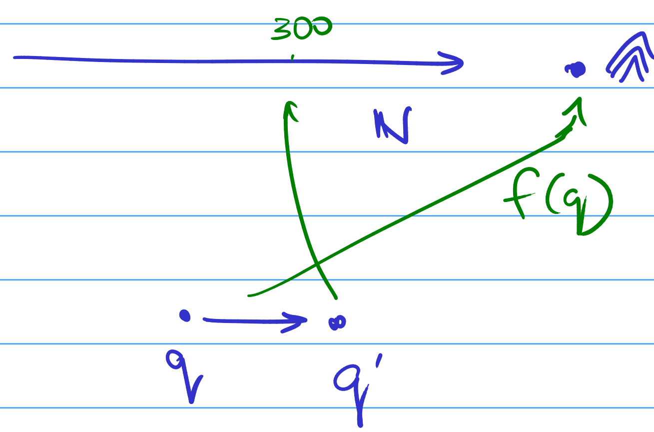
Observation: There is no order preserving bijection between $(\mathbb{N}, \leq)$ & $(\mathbb{N} \cup \{\infty\}, \leq)$.

$$1+\omega = \omega \neq \omega+1$$

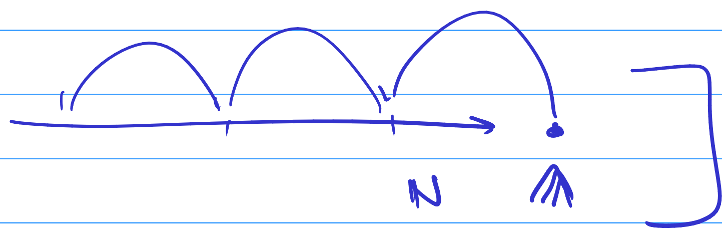
Proving program termination with $(\mathbb{N} \cup \{\Uparrow\})$

Say we create a function $f : Q \to \mathbb{N} \cup \{\Uparrow\}$ such that:

- $\forall q \; q' \quad q \to q' \Rightarrow f(q) \gneq f(q')$
- $\forall q \; q' \quad q \to q' \Rightarrow f(q) \gneq 0$.

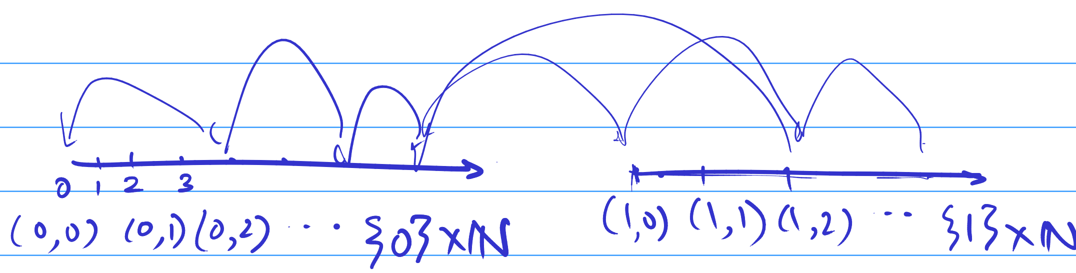


Claim: There are no infinite descending chains through $\mathbb{N} \cup \{\Lambda\}$. (where $\Lambda \gneq$ everything else)



$\omega + 1$

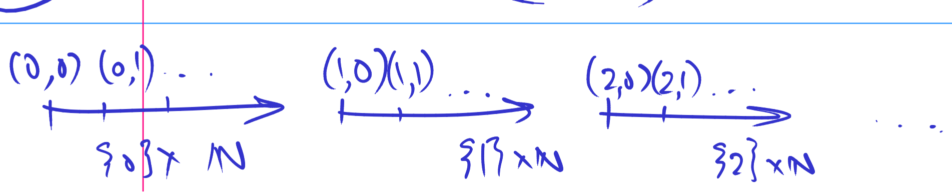
Claim: There are no infinite descending chains through $(\{0\} \times \mathbb{N}) \cup (\{1\} \times \mathbb{N})$



$$\omega + \omega \gneq \omega + 1 \gneq \omega = 1 + \omega$$

Claim: There are no infinite descending chains through $(\mathbb{N} \times \mathbb{N})$ $\quad (a, b) \leq (a', b')$ if $a \leq a'$ or if $a = a'$ & $b \leq b'$

ω^2



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