Lecture 22 Program Termination

- Well ordered set $(Q \quad \le)$

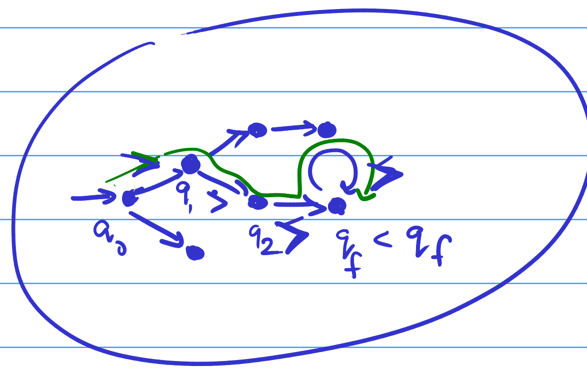
program states → Q; $\le$ ← contains the transition relation

Q ↑ elements; $\le$ ↑ total order

such that no infinite descending chains.

$$q_0 \gneq q_1 \gneq q_2 \gneq \cdots \infty$$

```
while (true) {
    skip
}
```



$$q_0 \gneq q_1 \gneq q_2 \gneq q_f \gneq q_f \gneq q_f \gneq \ldots$$

Claim: A program terminates iff there is a well order $(Q \quad \le)$ over its state space Q such that $\le$ is consistent with its transition relation.

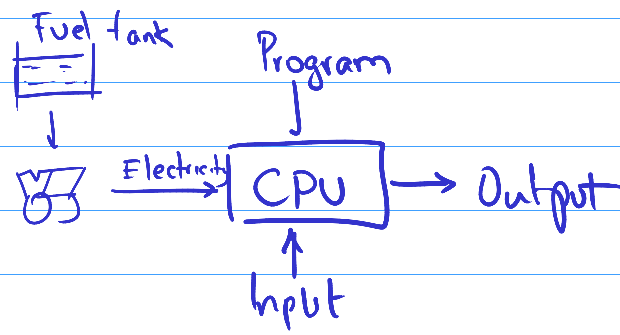
$$q \to q' \Rightarrow q \gneq q'$$

Simple ranking functions

$f: Q \to \mathbb{N}$

Given a state

How many steps before termination



Lexicographic Ranking Functions

A function of the form $f: Q \to \mathbb{N} \times \mathbb{N}$

second component

program state

first component

$$(a, b) \lneq (a', b') \text{ iff } a \lneq a' \text{ or}$$
$$\text{if } (a = a' \text{ and } b \lneq b')$$

Imagine a dictionary.

"ab" before "bb"

before "ba".

"aa" before "ab".

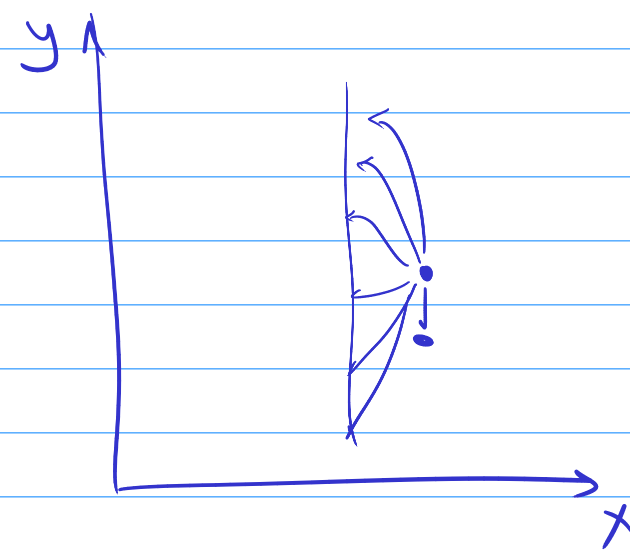
Claim: A program always terminates if its transition relation is consistent with some lexicographic ranking function.

$$f : Q \to \mathbb{N} \times \mathbb{N}.$$

Disjunctive Ranking Functions

Ex: ①

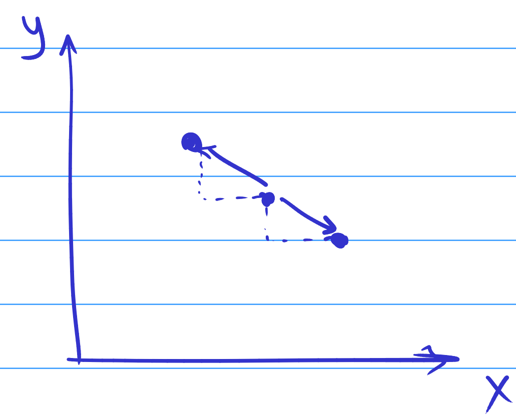
```
x := input
y := input
while x>0 and y>0
    if input()=1 then
        x := x-1
        y := input
    else: y := y-1
```



<u>Observation1</u>: In every iteration, either x goes down or y goes down.

Ex ②

```
x := input()
y := input()
while x > 0 and y > 0:
    if input() = 1 then
        x := x-1
        y := y+1
    else:
        x := x+1
        y := y-1
```



<u>Observation2</u>: Across any number of iterations, either x goes down or y goes down.

Ex① $\models$ Obs 2 Ex② $\not\models$ Obs 2

Claim (Podelski & Rybalchenko, LICS 2004)

A program is terminating iff it admits a disjunctively well-founded transition relation

Terminates iff $\exists (\leq_1, \leq_2 \cdots \leq_n)$

such that whenever $q_1 \to q_2 \to \cdots \to q_k$

either $(q_1 \leq_1 q_k)$ or

$(q_1 \leq_2 q_k)$ or

$\cdots$ or

$(q_1 \leq_n q_k)$

Proof: ($\Leftarrow$ case) Assume not. Say there is an infinite computation:

$q_0 \to q_1 \to q_2 \to \cdots \to \infty$

Pick every pair of states along this computation.

$$\begin{tikzcd} q_0 \arrow[r] & q_1 \arrow[r] \arrow[rrr, bend left, "2"] & q_2 \arrow[r] & \cdots \arrow[r] & q_{400} \arrow[r] & q_{401} \arrow[r] & \cdots \infty \end{tikzcd}$$

Either $q_1 \leq_1 q_{400}$

or $\boxed{q_1 \leq_2 q_{400}}$

or ...

or $q_1 \leq_n q_{400}$

<u>Ramsey's Theorem</u>: If you have an infinite clique, & a finite set of colours applied to its edges, then there is an infinite monochromatic subclique.