Lecture 4

Given: CNF formula φ

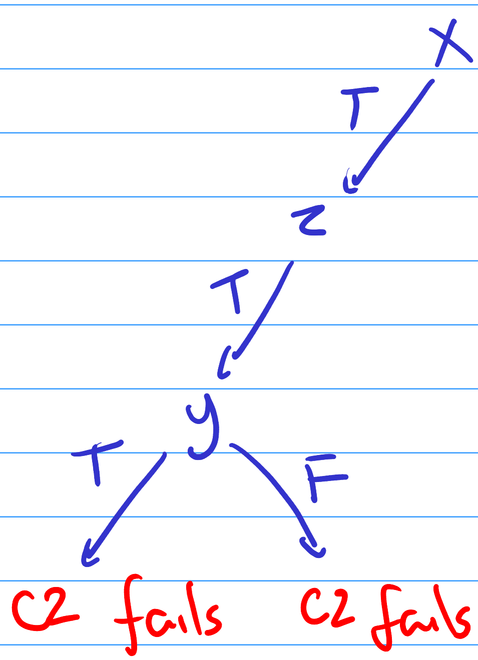
(___ or ___ or ___) and
(___ or ___) and
(___ or ___) and ...

To find: Is φ satisfiable?

Alg 1:

- If there is an unassigned variable x
- Check if $\varphi[x := \text{true}]$ is sat
  If yes, then SAT
- Otherwise, check if $\varphi[x := \text{false}]$ is sat
  If yes, then SAT
- Otherwise, UNSAT

Ex: $\varphi = (y \vee z \vee x)$ and
$(\overline{z} \vee \overline{x})$ and
$(y \vee \overline{z})$



$(\bar{z} \vee \cancel{\bar{x}})$

$\Downarrow \bar{z}$ $\qquad$ $\searrow T$

$x \xrightarrow{T}$ Drop literal $\bar{x}$

$x \xrightarrow{F}$ Drop the whole clause

$\varphi = (y \vee z \vee x)$ and $(\bar{z} \vee \bar{x})$ and $(y \vee \bar{z})$

$\downarrow$

$x \mapsto \text{True}$

$\downarrow$

$\varphi =$ Dropped clause $(\bar{z})$ and $(y \vee \bar{z})$

$\downarrow$

$z \mapsto \text{True}$

$\downarrow$

$\varphi =$ Dropped literal $(\quad)$ and $(y \vee \bar{z})$

Alg 2

1. If φ is empty, then sat
2. If an empty clause exists, then unsat
3. Pick an variable x

· Check if $\varphi[x := \text{true}]$ is sat or
$\varphi[x := \text{false}]$ is sat

Ex $\varphi = (\bar{x} \vee y)$ and $(\bar{x} \vee y \vee z)$ and $(\bar{x} \vee y \vee w)$

$y \xrightarrow{F}$

$\bar{x}$ and $(\bar{x} \vee z)$ and $(\bar{x} \vee w)$

$x \xrightarrow{F}$

$z \xrightarrow{F}$

$\bar{x}$ and $\bar{x}$ and $(\bar{x} \vee w)$

$w \xrightarrow{F}$

$\bar{x}$ and $\bar{x}$ and $\bar{x}$

$x \xrightarrow{T}$ ~~~ and ~~~ and ~~~

$x \xrightarrow{F}$ ✓

View 1 : There is a clause with exactly one literal. "Unit"

Set that literal to true.

"Unit propagation" / "Boolean constraint propagation"

UP (φ) := Walk over the clauses of φ

If clause contains a single lit, set it to true

Alg3 DPLL (φ)

Let $\varphi' = UP(\varphi)$

Do the same thing as Alg 2.

Quine — DP/DPLL — Bryant BDD

1952 1960/62 1986

~10 vars ~10 vars ~100 vars

CHAFF
Moskowitz
2001
Watched lists
VSIDS
Restarts
~10 K vars

GRASP
Marques Silva
Karem Sakallah
CDCL
1K vars

Polynomial time algorithms for Horn SAT & 2-SAT

$= 2\text{-SAT}$ $\qquad\qquad$ $\leq 2\text{-SAT}$

$$(x \text{ or } x) \text{ and } (y \text{ or } z) \xleftarrow{\text{UP}} x \text{ and } (y \text{ or } z)$$

Horn-CNF is a formula where every clause is a "Horn clause"

Horn-clause = At most one positive literal.

$$\bar{x} \vee \bar{y} \vee z \iff \overline{x \wedge y} \vee z$$

$$\Updownarrow$$

$$x \wedge y \Rightarrow z$$

Case 1 : No positive literals $\bar{x} \vee \bar{y}$

"Goals" $\overline{x \wedge y}$

Case 2 : One positive lit, some negative lits

"Definite clauses" $\bar{x} \vee \bar{y} \vee z$

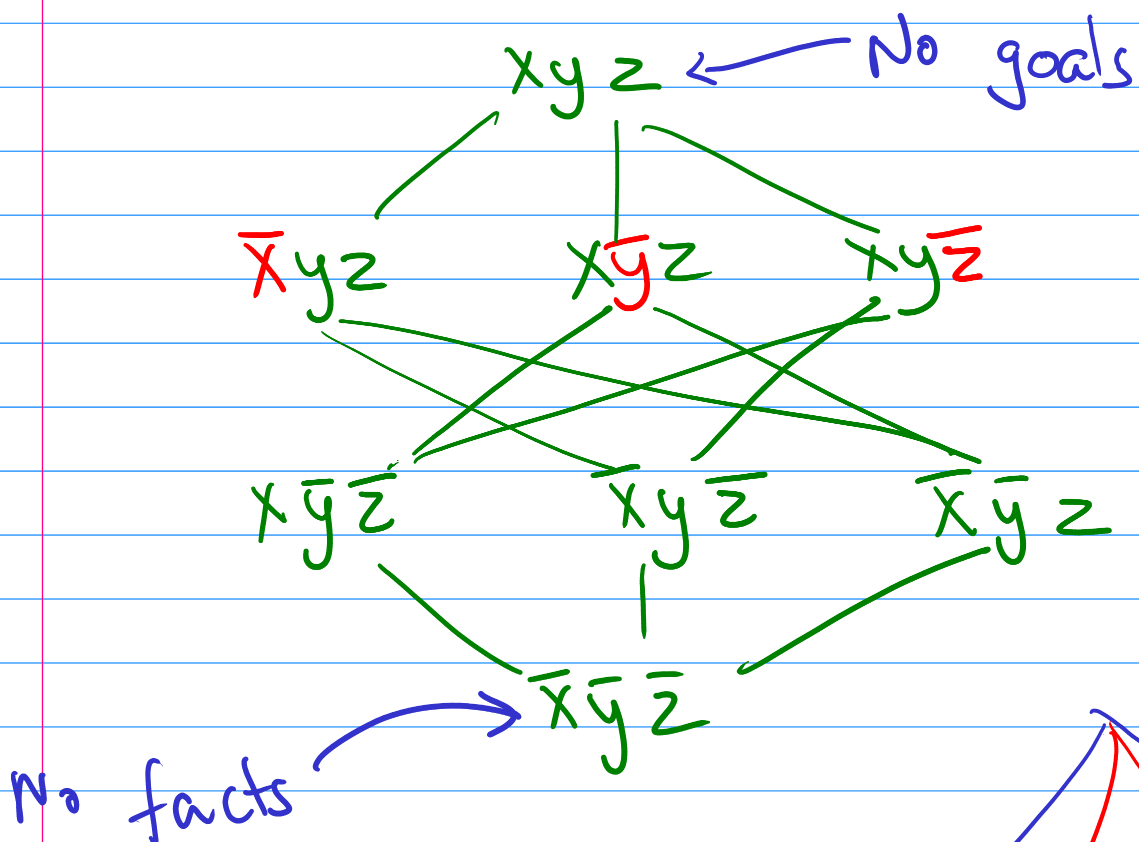
$x \wedge y \Rightarrow z$

Case 3 : One positive lit, no negative lits

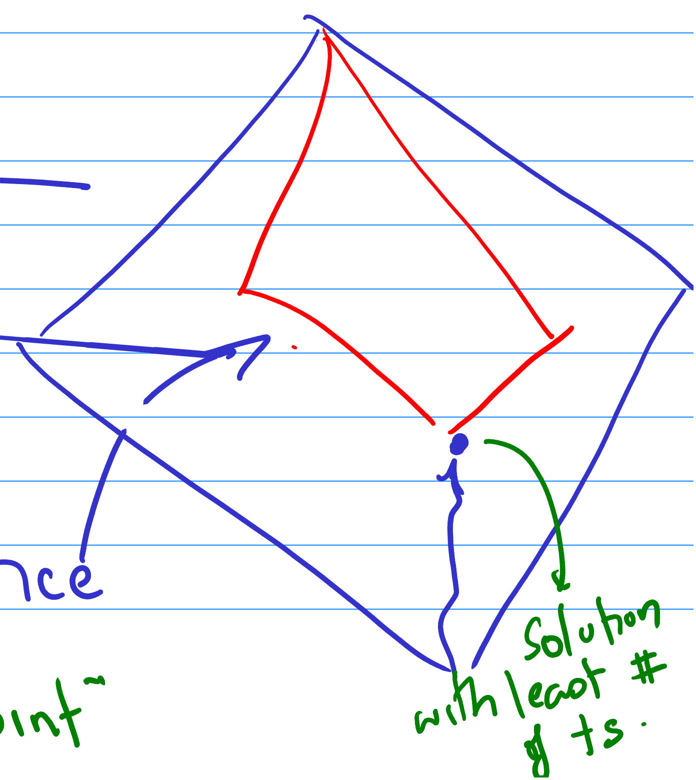
"Fact" y

Case 1 : No goals $\implies$ set everything to true

Case 2 : No facts $\implies$ set everything to false



As you do unit propagation, if the formula is sat soln lives in this sub-lattice



"Least fixpoint"

2-SAT

Every clause has exactly two literals

$$x \vee y \iff \bar{\bar{x}} \vee y \iff \bar{x} \Rightarrow y$$

$$\bar{y} \Rightarrow x$$

$y \Rightarrow x$

$\bar{x} \Rightarrow \bar{y}$

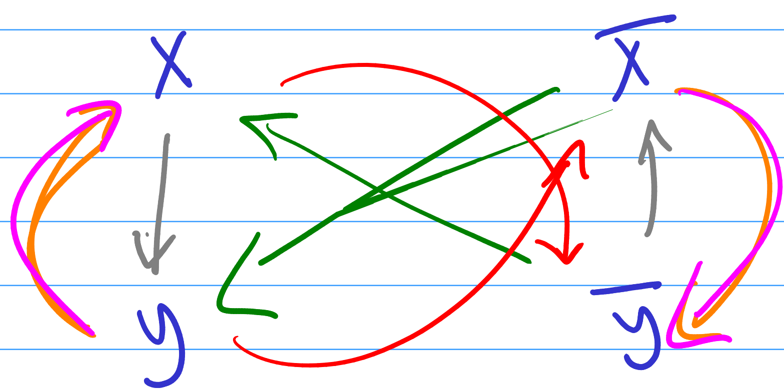
$\bar{y} \Rightarrow \bar{x}$

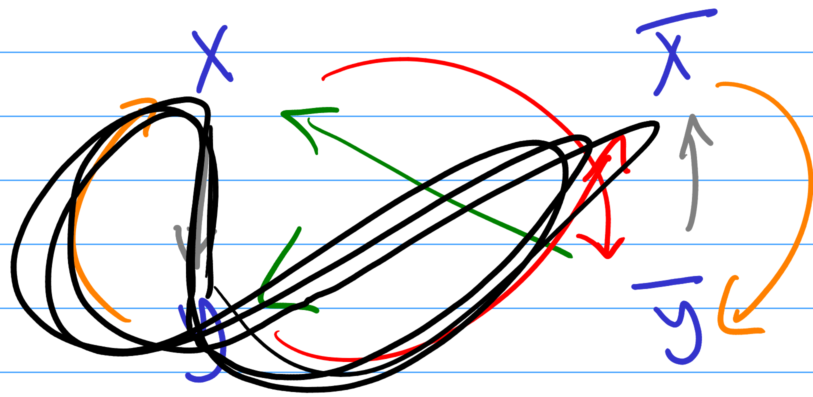
$x \Rightarrow y$

Ex $(x \vee \bar{y}) \,\&\, (\bar{x} \vee y) \,\&\, (\bar{x} \vee \bar{y}) \,\&\, (x \vee y)$

$x \Rightarrow \bar{y}$

$y \Rightarrow \bar{x}$





Bad SCC = Contains both a var & its negation

Claim: A 2-CNF formula φ is unsat iff its implication graph contains a bad SCC.

$\Leftarrow$: Obvious

$\Rightarrow$

Observation : Implication graph is skew symmetric

"Isomorphic to graph with reversed edges".