Theories

- LRA LIA RA IA
- Difference logic
- Theory of equality / EUF

$\exists x. \quad x + 2.5 \leq 3.8$ LRA

$\exists f. \ \exists x. y. \quad f(x) = f(y)$ and $x \neq y$ EUF

$\exists x. y. \quad x - y \leq 4$ and $y - x \leq 8$ IDL "non-logical"

Set of symbols Σ

Each symbol has an arity

"Signature".

~~"Logic"~~ + variables

+ "$=$" "logical"

+ "$\exists$", "$\forall$"

+ "$\vee, \wedge, \neg$"

"Sentences"

$2.0 + 4.8 = 6.8$

$3.3 - 2.6 = 1.9$

$3.3 \neq 4.8$

"Theory" is a set of sentences

$\exists x. \quad 3 < x$ and $x < 4$ LIA

LRA

Ex : $x = y \quad y = z \quad x \neq z$

$$x \text{ ———— } y \text{ ———— } z$$

Ex $x = y \quad y \neq a \quad a = b \quad a \neq c \quad c = d$

$d \neq y$

$$\begin{array}{ccc} x & a & c \\ | & | & | \\ y & b & d \end{array}$$

Equality with uninterpreted fns

Ex: $\exists f \;\; \exists x \, . \;\; f(x) = x$ and $f(f(x)) \neq f(x)$?

$$x \;\text{———}\; f(x) \;\sim\!\sim\!\sim\!\sim\; f(f(x))$$

<u>Ex</u> $\exists f \cdot \exists x\, y \cdot \ x = y$ but $f(x) \neq f(y)$

$$x \text{ ———— } y$$

$$f(x) \sim\!\sim\!\sim\!\sim f(y)$$

Term, $e ::= x \mid y \mid z \mid \cdots \mid f(e) \mid g(e) \mid h(e) \mid \cdots$

Equality constraints, $e_1 = e_2$

Ineq cts $e_1 \neq e_2$

Formula : $e_1 = e_2$ and $e_3 = e_4$ and ... and $e_k = e_{k+1}$ and $e_{k+2} \neq e_{k+3}$ and $e_{k+4} \neq e_{k+5}$ and ...

Congruence Closure

1. Let the vertices $V =$ all subterms in φ
2. For every equality constraint $e_1 = e_2$ in φ draw an $e_1 — e_2$ edge
3. Repeat until no progress possible:
   if I see $e_1 — e_2$ edge & $f(e_1)$ & $f(e_2) \in V$
   then draw $f(e_1) \rightsquigarrow f(e_2)$ edge

4. Check if any inequality constraint $e_1 \neq e_2$ relates vertices in the same SCC.

5. sat if no, unsat if yes.

Ex Defining evenness (MSO with 1-successor)

$$\text{even}(x) = \exists y.\ x = y + y$$

+1

$$\text{even}(x) = \forall E.\ (0 \in E \text{ and } \forall x.\ x \in E \Rightarrow x+2 \in E) \Rightarrow x \in E$$

- 0 is even
- x is even $\Rightarrow$ x+2 is even
- Smallest set with above properties

- DPLL(T) Theory
- Theory combination / Nelson-Oppen

Theory solver (T)

<u>Input</u>: Conjunctive Σ-formula φ

<u>Output</u>: Yes iff $\varphi \in T$, no otherwise

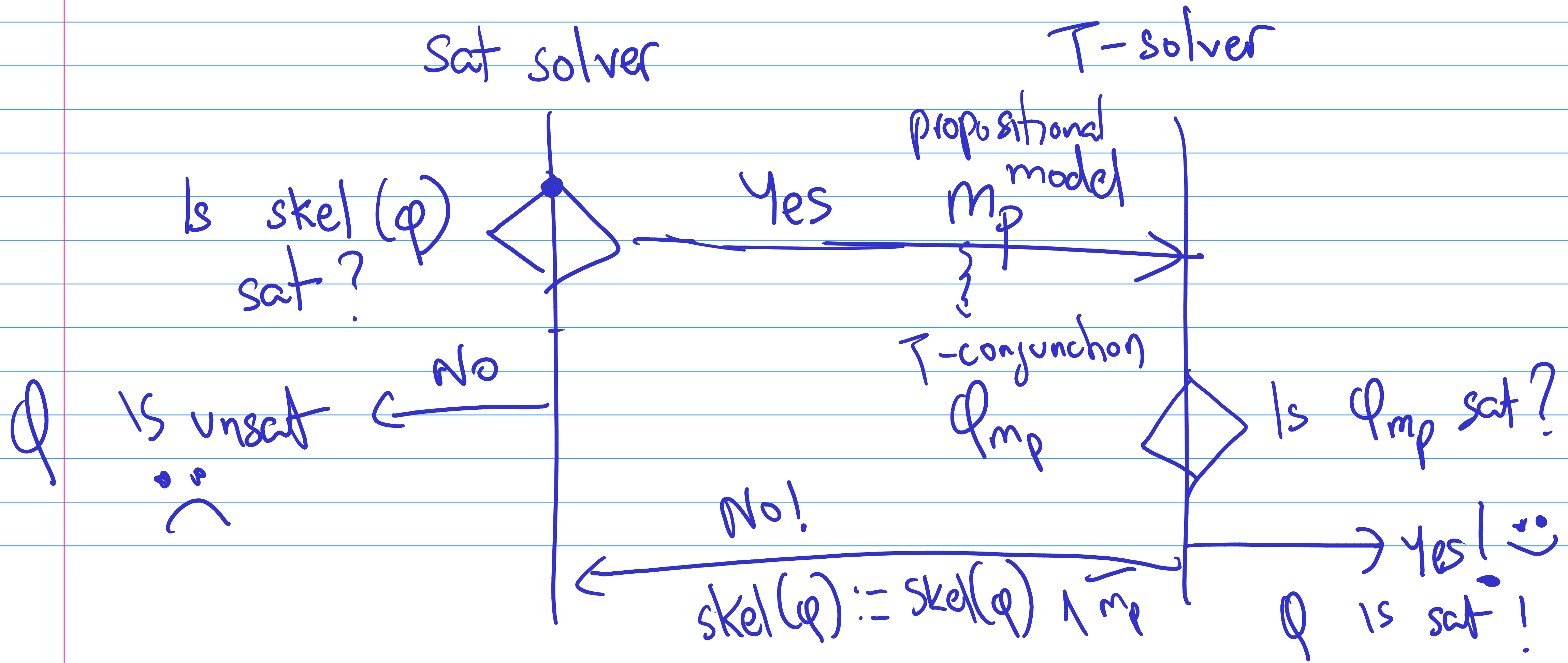
Ex: $x = y$ and $\left((y = z \text{ and } x \neq z) \text{ or } x = z \right)$

Propositional skeleton: v_1 and $\left((v_2 \text{ and } v_3) \text{ or } v_4 \right)$

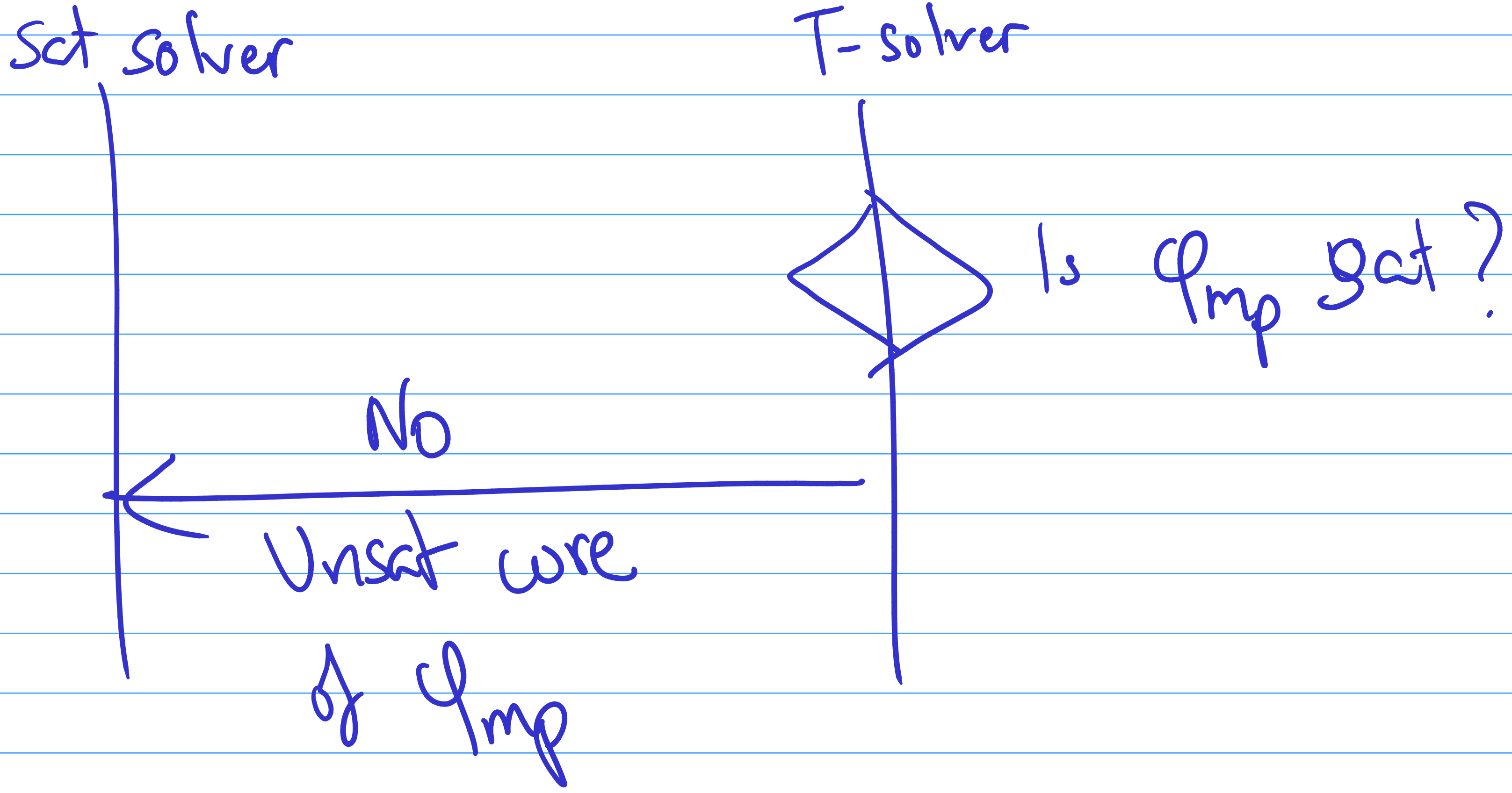
Propositional model: $v_1 \mapsto \text{true}, \; v_2 \mapsto \text{true}, \; v_3 \mapsto \text{true}, \; v_4 \mapsto \text{true}$

$x = y$ and $y = z$ and $x \neq z$ and $x = z$

DPLL(T) Version 1.0



DPLL(T) Version 2.0



DPLL (T) Version 3.0

Theory propagation: $\varphi \Rightarrow x = y$

or $\varphi \Rightarrow x \neq y$

Learn theory lemmas as the DPLL procedure proceeds.